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Abstract

Aspects of Poisson-Lie T-duality are reviewed in more algebraic way than in our, rather geometric, previous papers. As a new result, a moment map is constructed for the Poisson-Lie symmetry of the system consisting of open strings propagating in a Poisson-Lie group manifold.

1 INTRODUCTION

T-duality can be interpreted as the strong-weak coupling field theoretical duality from the world-sheet point of view but also as a discrete symmetry of the string theory from the space-time one [1]. In the paper [2] we have argued that the T-duality is in fact a manifestation of the well known duality in the category of Poisson-Lie groups at the classical level and of Hopf algebras (quantum groups) at the quantum level. We have shown how the structural features of the Abelian T-duality can be encoded in the language of Poisson-Lie groups, Drinfeld doubles, Lie bialgebras, Manin triples etc. We have argued that this language, though it is not really necessary tool for studying the Abelian case, can be directly used for the non-Abelian generalizations of the T-duality where it becomes essential.

In a sense, we kept stressing the stringy applications of the formalism in our previous works because we personally discovered the rich Poisson-Lie world as a suitable

tool for handling problems arising in string research. In the course of developping the Poisson-Lie T-duality program we have often settled down our own terminology. This attitude looked quite safe since, to our best knowledge, the Poisson-Lie groups have not been applied previously in string theory. But after having got acquainted better with the theory of integrable models, we found the Poisson-Lie world to be an extremely interesting structure per se, with a well developped terminology. In particular, our notion of the ‘Poisson-Lie symmetry’ of a σ -model which we used in [2] is in clash with the notion of the Poisson-Lie symmetry of a general dynamical system in the sense of [3]. Moreover, our notion played an essential role in our formalism because the Poisson-Lie symmetry was *the* property required from a σ -model in order the duality transformation on it could be performed. We therefore felt a need to clarify our string results in the form appropriate for Poisson-Lie experts. Having performed this exercise turned out to be fruitful not only from the terminological point of view; in fact, we have discovered the traditional Poisson-Lie symmetry of our ‘Poisson-Lie’ symmetric σ -models. Thus both notions are intimately connected and we are devoting this article to a detailed description of this connection. As often before, it turns out that an elegant and well-understood structure in the Poisson-Lie world finds a natural manifestation in the world of the Poisson-Lie T-dualizable σ -models.

In the second (third) section we describe our (traditional) notion of the Poisson-Lie symmetry. We characterize the interplay between the two in section 4 where also a moment map of the traditional Poisson-Lie symmetry is constructed for the case of open strings in the backgrounds possessing the ‘new’ Poisson-Lie symmetry.

2 “NEW” POISSON-LIE SYMMETRY of σ -MODELS

Consider a $2n$ -dimensional group D such that its Lie algebra \mathcal{D} (viewed as a vector space) can be decomposed as the direct sum of two subalgebras, \mathcal{G} and $\tilde{\mathcal{G}}$, maximally isotropic with respect to a non-degenerate invariant bilinear form on \mathcal{D} [3]. Such a group D is referred to as the Drinfeld double. If, moreover, each element of D can be uniquely written as the product of two elements of the two groups G and \tilde{G} in both possible orders of G and \tilde{G} we shall refer to D as to the ‘perfect’ Drinfeld double. Of course \mathcal{G} and $\tilde{\mathcal{G}}$ are the Lie algebras of G and \tilde{G} , respectively and it is often said that the groups G and \tilde{G} form the double D . Throughout this article, we shall work with the perfect doubles.

There exists a natural symplectic structure on the group manifold D , first introduced by Semenov-Tian-Shansky in [4]. It will play the crucial role in our presentation, therefore we devote some place to the description of its properties. For doing that, define $(\nabla_L f)_a, (\nabla_L f)^a, (\nabla_R f)_a$ and $(\nabla_R f)^a$ as

$$\begin{aligned} df &= (\nabla_L f)_a (dll^{-1})^a + (\nabla_L f)^a (dl^{-1})_a = \\ &= (\nabla_R f)_a (l^{-1}dl)^a + (\nabla_R f)^a (l^{-1}dl)_a, \end{aligned} \quad (1)$$

where f is some function on the double and $l \in D$ parametrizes the group manifold D . Clearly, the upper and lower indices for the forms dll^{-1} (or $l^{-1}dl$) mean

$$dll^{-1} = (dll^{-1})_a T^a + (dll^{-1})^a \tilde{T}_a \quad (2)$$

and T^a and \tilde{T}_a are the generators of \mathcal{G} and $\tilde{\mathcal{G}}$, respectively, satisfying the duality property

$$\langle T^a, \tilde{T}_b \rangle = \delta_b^a. \quad (3)$$

Needless to say, in all formulas $\langle \cdot, \cdot \rangle$ denotes the invariant bilinear form on the double. Then the Semenov-Tian-Shansky Poisson bracket is given by

$$\{f, f'\}_D = (\nabla_L f)_a (\nabla_L f')^a - (\nabla_R f)^a (\nabla_R f')_a \quad (4)$$

for arbitrary functions f, f' on the double.

Consider the functions f, f' in (4) to be invariant with respect to the right action of the group \tilde{G} (G) on D . Then they can be interpreted as functions on the group manifold G (\tilde{G}) and their Poisson bracket (4) defines a Poisson bracket on the group manifold G (\tilde{G}). This Poisson bracket can be written as

$$\{f, f'\}_G = \Pi_{ab}(g) (\nabla_L f)_a (\nabla_L f')_b, \quad (5)$$

where $\Pi(g)$ is certain antisymmetric tensor field on G whose explicit form can be easily derived from (4). It is given by

$$\Pi(g) = b(g)a(g)^{-1}, \quad (6)$$

where

$$\langle g^{-1}T^i g, \tilde{T}_j \rangle \equiv a(g)^i_j, \quad \langle g^{-1}\tilde{T}_i g, \tilde{T}_j \rangle \equiv b(g)_{ij}. \quad (7)$$

The derivatives $(\nabla_L f)^a$ and $(\nabla_L f')^b$ in (5) are defined with respect to the group manifold G . Of course, the role of the groups G and \tilde{G} can be interchanged and, up to a sign*, we obtain an exactly corresponding Poisson bracket $\tilde{\Pi}^{ab}(\tilde{g})$ on \tilde{G} . A pair of groups G and \tilde{G} equipped with the Poisson brackets $\Pi(g)$ and $\Pi(\tilde{g})$, respectively, is called a dual pair of the Poisson-Lie groups.

In [2], we have constructed a dual pair of σ -models[†] on the group manifolds G and \tilde{G} . Their Lagrangians were respectively given as follows

$$L = E(g)^{ab} (\partial_+ g g^{-1})_a (\partial_- g g^{-1})_b; \quad (8a)$$

$$\tilde{L} = \tilde{E}(\tilde{g})_{ab} (\partial_+ \tilde{g} \tilde{g}^{-1})^a (\partial_- \tilde{g} \tilde{g}^{-1})^b, \quad (8b)$$

where

$$\partial_{\pm} \equiv \partial_{\tau} \pm \partial_{\sigma} \quad (9)$$

and

$$E^{-1}(g)_{ab} = R_{ab}^{-1} + \Pi_{ab}(g), \quad \tilde{E}^{-1}(\tilde{g})^{ab} = R^{ab} + \tilde{\Pi}^{ab}(\tilde{g}). \quad (10)$$

Here R is an arbitrary non-degenerate matrix. We have found in [2], that both models (8ab) are ‘Poisson-Lie symmetric’ in the following sense:

Definition: A σ -model on a Poisson-Lie group manifold G is called Poisson-Lie symmetric if the Noether current one-forms $\tilde{J}(g) \in \tilde{\mathcal{G}}$ fulfil the zero-curvature condition

$$d\tilde{J}(g) - \tilde{J}(g)^2 = 0, \quad (11)$$

for every solution g of the σ -model field equations. Recall that the Noether current one-forms $\tilde{J}(g)$ are defined by the variation of the σ -model action with respect to the right action of G on itself

$$\delta \int L = \int \langle \tilde{J}(g) \wedge d\epsilon \rangle + \int \epsilon^a \mathcal{L}_{v_a}(L,) \quad (12)$$

where $g + \delta g = g(1 + \epsilon)$, $\epsilon \in \mathcal{G}$ and $\mathcal{L}_{v_a}(L)$ are the Lie derivatives of the Lagrangian (see [2] for more details).

Note that the models (8a) and (8b) are both Poisson-Lie symmetric; the role of the groups G and \tilde{G} in passing from (8a) to (8b) gets interchanged.

*The Poisson structure (4) changes the sign upon exchanging G and \tilde{G} (cf. (28)).

†These models were shown to be dynamically equivalent (hence dual) in [2].

3 TRADITIONAL POISSON-LIE SYMMETRY

Suppose we are given a manifold P with the Poisson structure π ($\pi \in \wedge^2 TP$) and a right action $a : P \times G \rightarrow P$ of G on P , generated by a section v of the bundle $TP \otimes \mathcal{G}^* = TP \otimes \tilde{\mathcal{G}}$. If we can find a map $\tilde{m} : P \rightarrow \tilde{G}$ such that

$$v = \pi(d\tilde{m}\tilde{m}^{-1}) \quad (13)$$

and \tilde{m} is equivariant with respect to the action a on P and the dressing action of G on \tilde{G} then \tilde{m} is called the moment map of the Poisson-Lie action a of the Poisson-Lie group G on the Poisson manifold P . Note that on this section we inheritate the terminology of the previous one: G and \tilde{G} form the dual pair of the Poisson-Lie groups; moreover, we understand that the dual spaces \mathcal{G}^* and $\tilde{\mathcal{G}}^*$ are naturally identified with $\tilde{\mathcal{G}}$ and \mathcal{G} , respectively, via the bilinear form $\langle \cdot, \cdot \rangle$ on the double. Recall also that the definition of the dressing action of G on \tilde{G} :

$$\tilde{g}h = g\tilde{h}, \quad g, h \in G, \quad \tilde{g}, \tilde{h} \in \tilde{G}. \quad (14)$$

Here h acts on \tilde{g} and the result of the action is \tilde{h} . The fact that the element of the double $\tilde{g}h$ can be uniquely represented as the product $g\tilde{h}$ follows from our assumption that the Drinfeld double is perfect.

As an example of the moment map for a Poisson-Lie action, consider the Drinfeld double D itself as the manifold P ; the Poisson structure π is given by the Semenov-Tian-Shansky bracket (4). The action of the Poisson-Lie group G on D is given simply by right multiplication of the elements of D by the elements of its subgroup G . The moment map for this action is given as

$$\tilde{m}(l) = \tilde{h}, \quad l \in D, \quad (15)$$

where \tilde{h} is given by the following decomposition of the arbitrary element $l \in D$:

$$l = g\tilde{h}, \quad g \in G, \quad \tilde{h} \in \tilde{G}. \quad (16)$$

The fact that \tilde{h} is the moment map of this Poisson-Lie action can be easily checked by direct computation (see also [4]). Note, that the role of the groups G and \tilde{G} can be interchanged and the right action of \tilde{G} on D is also the Poisson-Lie action whose moment map can be constructed in the exactly corresponding way.

If the manifold P is the symplectic manifold, which means that the bivector field π can be inverted to give the symplectic form ω on P , then, as the corollary of (13), we have the relation

$$i_v \omega = d\tilde{m}\tilde{m}^{-1}. \quad (17)$$

Here i_v means the contraction of the form by the vector field. We say that a dynamical system, whose phase space is the symplectic manifold (P, ω) is Poisson-Lie symmetric with respect to the group G if

$$\mathcal{L}_v H = 0, \quad (18)$$

where H is the Hamiltonian of the system and $v \in TP \otimes \tilde{\mathcal{G}}$ generates the Poisson-Lie action of G on P .

4 A COMPARISON

Consider an open string in the Poisson-Lie group manifold G whose propagation is governed by (8a). The open string boundary conditions at the world sheet boundaries $\sigma = 0, \pi$ read

$$\partial_\sigma g g^{-1}|_{0,\pi} = 0. \quad (19)$$

They are usually referred to as the von Neumann conditions and they insure that there is no momentum flow through the boundary of the string. But here the ‘momentum’ \tilde{m} is a group valued quantity defined for every extremal world sheet (=a point in the phase space of the system) by calculating the path-ordered integral of the Noether current $\tilde{J}(g)$ over a path γ starting at one edge of the open string world sheet and ending at the other edge:

$$\tilde{m} = P \exp \int_\gamma \tilde{J}(g). \quad (20)$$

Note that the momentum \tilde{m} is the conserved quantity; it does not depend on the path γ (in particular on the time in which γ crosses the world sheet) by virtue of the equation (11) and the boundary conditions (19). Of course, we have a good reason to denote the \tilde{G} -valued momentum as \tilde{m} ; it is going to be precisely the moment map (from the phase space of the system (8a) into the group \tilde{G}) that generates the *traditional* Poisson-Lie symmetry of the model (8a). In order to prove this, it is convenient to write the action of the model (8a) in the first-order (Hamiltonian) form [5]

$$S = \int L_H, \quad (21)$$

where

$$L_H = \langle \tilde{\Lambda}, g^{-1} \partial_\tau g \rangle - \frac{1}{2} Ad_g G(\tilde{\Lambda}, \tilde{\Lambda}) - \frac{1}{2} Ad_g G^{-1}(g^{-1} \partial_\sigma g + Ad_g(B + \Pi(g))(\cdot, \tilde{\Lambda}), g^{-1} \partial_\sigma + Ad_g(B + \Pi(g))(\cdot, \tilde{\Lambda})). \quad (22)$$

Here $\tilde{\Lambda}$ is the canonically conjugated momentum that is naturally valued in $\mathcal{G}^* \equiv \tilde{\mathcal{G}}$. The bracket $\langle \cdot, \cdot \rangle$ can therefore be understood in two ways: either as the standard pairing between algebra and coalgebra or as the bilinear form in the Lie algebra \mathcal{D} of the double D . We have used a compact notation in (22) in order not to burden the formula with too many indices: $G(\cdot, \cdot)$ and $B(\cdot, \cdot)$ are the symmetric and the antisymmetric part, respectively, of the bilinear form $R^{-1}(\tilde{T}_a, \tilde{T}_b) = (R^{-1})_{ab}$; $G^{-1}(\cdot, \cdot)$ is, in turn, the inverse bilinear form to $G(\cdot, \cdot)$ and, as such, it is defined on the Lie algebra \mathcal{G} . Ad_g means the adjoint action of the group G on the bilinear forms.

Our crucial trick is the following: we parametrize the canonically conjugated momentum $\tilde{\Lambda}$ by a field \tilde{h} (valued in \tilde{G}) such that

$$\tilde{\Lambda} = \partial_\sigma \tilde{h} \tilde{h}^{-1}. \quad (23)$$

The ambiguity of this parametrization is fixed by requiring that

$$\tilde{h}(\tau, \sigma = 0) = \tilde{e}, \quad (24)$$

where \tilde{e} is the unit element of \tilde{G} . The first order Lagrangian then can be rewritten as

$$L_H = \langle \partial_\sigma \tilde{h} \tilde{h}^{-1}, g^{-1} \partial_\tau g \rangle - \frac{1}{2} \langle \partial_\sigma l \, l^{-1}, A \partial_\sigma l \, l^{-1} \rangle, \quad (25)$$

where $l = g\tilde{h}$ and A is a linear (idempotent) self-adjoint map from the Lie algebra \mathcal{D} of the double into itself. It has two eigenvalues $+1$ and -1 , the corresponding eigenspaces \mathcal{R}_+ and \mathcal{R}_- have the same dimension $\dim G$, they are perpendicular to each other in the sense of the invariant form on the double and they are given by the following recipe:

$$\mathcal{R}_+ = \text{Span}\{T^a + R^{ab}\tilde{T}_b\}; \quad (26a)$$

$$\mathcal{R}_- = \text{Span}\{T^a - R^{ba}\tilde{T}_b\}. \quad (26b)$$

The second term in the right hand side of (25) is (minus) Hamiltonian. Obviously, the phase space of our model is described by the functions g and \tilde{h} , satisfying the boundary conditions (19) and (24). Now we define an action of the group G on the phase space as follows

$$g\tilde{h}g_0 = g'\tilde{h}', \quad (27)$$

where $g_0 \in G$ acts and the pair g', \tilde{h}' is the result of the action. We immediately notice that this action respects the boundary conditions to be fulfilled by g and \tilde{h} . Moreover, the Hamiltonian is invariant for it can be written just as the function of $\partial_\sigma l l^{-1}$ where $l = g\tilde{h}$. Does this action of G on P give rise to the Poisson-Lie symmetry of our open string model in the traditional sense of this notion? The answer is affirmative; in order to prove this, we have to exploit again the properties of the Semenov-Tian-Shansky symplectic structure on the double D .

We have proved in [2], that the Semenov-Tian-Shansky symplectic form ω on D can be conveniently expressed as

$$\omega = \langle d\tilde{h} \tilde{h}^{-1} \wedge g^{-1}dg \rangle - \langle dh h^{-1} \wedge \tilde{g}^{-1}d\tilde{g} \rangle, \quad (28)$$

where we have used the following two parametrization of the group manifold D :

$$l = g\tilde{h}, \quad l \in D, \quad g \in G, \quad \tilde{h} \in \tilde{G}; \quad (29a)$$

$$l = \tilde{g}h, \quad l \in D, \quad \tilde{g} \in \tilde{G}, \quad h \in G. \quad (29b)$$

We already know from the previous section that if $v \in TP \otimes \tilde{\mathcal{G}}$ describes the infinitesimal right action of G on D then it holds

$$i_v\omega = d\tilde{h}\tilde{h}^{-1}, \quad l = g\tilde{h}. \quad (30)$$

Returning to our Hamiltonian first-order action (25), we observe that the first term in it can be conveniently rewritten as

$$\int_\rho d\tau d\sigma \langle \partial_\sigma \tilde{h}\tilde{h}^{-1}, g^{-1}\partial_\tau g \rangle = \int_\rho d\tau d\sigma \langle \partial_\sigma h h^{-1}, \tilde{g}^{-1}\partial_\tau \tilde{g} \rangle + \int_\rho {}^*\omega. \quad (31)$$

Here the ${}^*\omega$ means the pull-back of the Semenov-Tian-Shansky form on the world-sheet ρ of the open string and we used both parametrizations (29a) and (29b) of the double. Now it is obvious that under the action v of the group G the variation of this quantity becomes

$$\delta \int d\tau d\sigma \langle \partial_\sigma \tilde{h}\tilde{h}^{-1}, g^{-1}\partial_\tau g \rangle = \delta \int {}^*\omega \equiv \int {}^*\mathcal{L}_v\omega. \quad (32)$$

Because $\mathcal{L}_v = i_v d + di_v$ and ω is closed, we have

$$\delta \int d\tau d\sigma \langle \partial_\sigma \tilde{h}\tilde{h}^{-1}, g^{-1}\partial_\tau g \rangle = \int d {}^*i_v\omega = \int_{\partial\rho} d\tilde{h}\tilde{h}^{-1} =$$

$$= \int_{\tau=\tau_f} d\sigma \partial_\sigma \tilde{h} \tilde{h}^{-1} - \int_{\tau=\tau_i} d\sigma \partial_\sigma \tilde{h} \tilde{h}^{-1} + \int_{\sigma=\pi} d\tau \partial_\tau \tilde{h} \tilde{h}^{-1}. \quad (33)$$

Here $\tau_i(\tau_f)$ is some constant initial (final) time and the integral along the edge $\sigma = 0$ vanishes because of (24). The formula (33) is almost what we want. The reason is that the Hamiltonian first-order action is always of the form

$$S = \int_\gamma \alpha - H dt, \quad (34)$$

where α is the polarization form of the system (this means that $d\alpha = \Omega$ and Ω is the symplectic form on the phase space of the dynamical system) and γ is a path in the phase space. In our case

$$\int_\gamma \alpha = \int_\rho d\tau d\sigma \langle \partial_\sigma \tilde{h} \tilde{h}^{-1}, g^{-1} \partial_\tau g \rangle. \quad (35)$$

If we want to show that the action (27) of G on P is the traditional Poisson-Lie symmetry (the invariance of the Hamiltonian we have already shown), we have to prove that

$$i_v \Omega = d\tilde{m} \tilde{m}^{-1} \quad (36)$$

for some \tilde{G} -valued function \tilde{m} on the phase space P . Note that

$$i_v \Omega = i_v d\alpha = \mathcal{L}_v \alpha - d(i_v \alpha) \quad (37)$$

hence

$$\int_\gamma i_v \Omega = \delta \int_\rho d\tau d\sigma \langle \partial_\sigma \tilde{h} \tilde{h}^{-1}, g^{-1} \partial_\tau g \rangle - i_v \alpha|_{\tau_i}^{\tau_f}. \quad (38)$$

Here $\tau_{i(f)}$ is the initial (final) time of the path γ . By using the equation (33), we obtain

$$i_v \Omega = d\tilde{h}(\sigma = \pi) \tilde{h}^{-1}(\sigma = \pi), \quad (39)$$

hence the moment map \tilde{m} is just

$$\tilde{m} = \tilde{h}(\sigma = \pi). \quad (40)$$

Finally, we observe

$$\tilde{m} = P \exp \int_{\tau=const} \tilde{\Lambda} = P \exp \int_{\tau=const} \tilde{J}(g). \quad (41)$$

Thus, indeed, the \tilde{G} valued momentum \tilde{m} , introduced in (20), is the moment map of the traditional Poisson-Lie symmetry of our open string σ -model (8a).

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6 REFERENCES

1. A. Giveon, M. Porrati and E. Rabinovici, Phys. Rep. 244 (1994) 77
2. C. Klimčík and P. Ševera, Phys. Lett. B351 (1995) 455, hep-th/9502122; P. Ševera, *Minimálne Plochy a Dualita*, Diploma thesis, Praha University, May 1995 (in Slovak); C. Klimčík, Nucl. Phys. (Proc. Suppl.) 46 (1996) 116
3. F. Falceto and K. Gawędzki, J. Geom. Phys. 11 (1993) 251; A.Yu. Alekseev and A.Z. Malkin, Commun. Math. Phys. 162 (1994) 147
4. M. A. Semenov-Tian-Shansky, Publ. RIMS, Kyoto Univ. 21 (1985) 1237
5. C. Klimčík and P. Ševera, Phys. Lett. B372 (1996) 65